Université de Picardie Jules Verne

UFR d'économie et de gestion

Mathématiques

Licence 1 - Semestre 2

Exercices d'entrainement **Fonctions de deux variables (1/3)** **Corrigés**

Correction 1

1. Il faut que $x + y + 1 \geq 0$ et $x + y + 1 \neq 0$ c'est-à-dire $x + y + 1 > 0$.
   D'où $D_f = \{(x, y) \in \mathbb{R}^2 / x + y + 1 > 0\}$.
   Géométriquement parlant, il s'agit d'un demi-plan de frontière la droite d'équation $x + y + 1 = 0$.

2. Il faut que $xy \neq 0$. D'où $D_g = \{(x, y) \in \mathbb{R}^2 / x \neq 0 \text{ et } y \neq 0\}$.

3. Il faut que $1 - x^2 \geq 0$ c'est-à-dire $-1 \leq x \leq 1$ et que $x + 2y - 1 \neq 0$.
   D'où $D_h = \{(x, y) \in \mathbb{R}^2 / -1 \leq x \leq 1 \text{ et } x + 2y - 1 \neq 0\}$.

4. $D_i = \{(x, y) \in \mathbb{R}^2 / 2x + y - 3 > 0\}$.

Correction 2

1. $f : (x, y) \mapsto x^3y + 3xy^2 - x^2 + 3y + 5$
   $\dfrac{\partial f}{\partial x}(x, y) = f'_x(x, y) = 3x^2y + 3y^2 - 2x.$
   $\dfrac{\partial f}{\partial y}(x, y) = f'_y(x, y) = x^3 + 6xy + 3.$
   $\dfrac{\partial^2 f}{\partial x^2}(x, y) = f''_{x^2}(x, y) = 6xy - 2.$
   $\dfrac{\partial^2 f}{\partial y \partial x}(x, y) = f''_{yx}(x, y) = 3x^2 + 6y.$
   $\dfrac{\partial^2 f}{\partial y^2}(x, y) = f''_{y^2}(x, y) = 6x.$
   $\dfrac{\partial^2 f}{\partial x \partial y}(x, y) = f''_{xy}(x, y) = 3x^2 + 6y.$

2. $g : (x, y) \mapsto \ln(x^2 + y - 2)$
   $\dfrac{\partial g}{\partial x}(x, y) = g'_x(x, y) = \dfrac{2x}{x^2 + y - 2}.$
   $\dfrac{\partial g}{\partial y}(x, y) = g'_y(x, y) = \dfrac{1}{x^2 + y - 2}.$

$$\frac{\partial^2 g}{\partial x^2}(x,y) = g''_{x^2}(x,y) = \frac{2\times(x^2+y-2) - 2x\times 2x}{(x^2+y-2)^2} = \frac{-2x^2+2y-4}{(x^2+y-2)^2}.$$
$$\frac{\partial^2 g}{\partial y\partial x}(x,y) = g''_{yx}(x,y) = -\frac{2x}{(x^2+y-2)^2}.$$
$$\frac{\partial^2 g}{\partial y^2}(x,y) = g''_{y^2}(x,y) = -\frac{1}{(x^2+y-2)^2}.$$
$$\frac{\partial^2 g}{\partial x\partial y}(x,y) = g''_{xy}(x,y) = -\frac{2x}{(x^2+y-2)^2}.$$

3. $h : (x,y) \mapsto (x^2-y)e^{xy}$

$$\frac{\partial h}{\partial x}(x,y) = h'_x(x,y) = 2xe^{xy} + (x^2y-y^2)e^{xy} = (x^2y-y^2+2x)e^{xy}.$$
$$\frac{\partial h}{\partial y}(x,y) = h'_y(x,y) = (-1)e^{xy} + (x^3-xy)e^{xy} = (x^3-xy-1)e^{xy}.$$
$$\frac{\partial^2 h}{\partial x^2}(x,y) = h''_{x^2}(x,y) = (2xy+2)e^{xy} + (x^2y^2-y^3+2xy)e^{xy} = (x^2y^2-y^3+4xy+2)e^{xy}.$$
$$\frac{\partial^2 h}{\partial y\partial x}(x,y) = h''_{yx}(x,y) = (x^2-2y)e^{xy} + (x^3y-y^2x+2x^2)e^{xy} = (x^3y-xy^2+3x^2-2y)e^{xy}.$$
$$\frac{\partial^2 h}{\partial y^2}(x,y) = h''_{y^2}(x,y) = (-x)e^{xy} + (x^4-x^2y-x)e^{xy} = (x^4-x^2y-2x)e^{xy}.$$
$$\frac{\partial^2 h}{\partial x\partial y}(x,y) = h''_{xy}(x,y) = (3x^2-y)e^{xy} + (x^3y-xy^2-y)e^{xy} = (x^3y-xy^2+3x^2-2y)e^{xy}.$$

4. $i : (x,y) \mapsto \dfrac{x^2+y^2}{x^2-y^2}$

$$\frac{\partial i}{\partial x}(x,y) = i'_x(x,y) = \frac{2x(x^2-y^2) - 2x(x^2+y^2)}{(x^2-y^2)^2} = \frac{-4xy^2}{(x^2-y^2)^2}.$$
$$\frac{\partial i}{\partial y}(x,y) = i'_y(x,y) = \frac{2y(x^2-y^2) - (-2y)(x^2+y^2)}{(x^2-y^2)^2} = \frac{4x^2y}{(x^2-y^2)^2}.$$
$$\frac{\partial^2 i}{\partial x^2}(x,y) = i''_{x^2}(x,y) = \frac{(-4y^2)(x^2-y^2)^2 - (-4xy^2)(4x)(x^2-y^2)}{(x^2-y^2)^4} = \frac{12x^2y^2+4y^4}{(x^2-y^2)^3}.$$
$$\frac{\partial^2 i}{\partial y\partial x}(x,y) = i''_{yx}(x,y) = \frac{(-8xy)(x^2-y^2)^2 - (-4xy^2)(-4y)(x^2-y^2)}{(x^2-y^2)^4} = \frac{(-8xy)(x^2+y^2)}{(x^2-y^2)^3}.$$
$$\frac{\partial^2 i}{\partial y^2}(x,y) = i''_{y^2}(x,y) = \frac{(4x^2)(x^2-y^2)^2 - (4x^2y)(-4y)(x^2-y^2)}{(x^2-y^2)^4} = \frac{4x^4+12x^2y^2}{(x^2-y^2)^3}.$$
$$\frac{\partial^2 i}{\partial x\partial y}(x,y) = i''_{xy}(x,y) = \frac{8xy(x^2-y^2)^2 - (4x^2y)(4x)(x^2-y^2)}{(x^2-y^2)^4} = \frac{(-8xy)(x^2+y^2)}{(x^2-y^2)^3}.$$

5. $j : (x,y) \mapsto \sin(x^2-3xy)$

$$\frac{\partial j}{\partial x}(x,y) = j'_x(x,y) = (2x-3y)\cos(x^2-3xy).$$
$$\frac{\partial j}{\partial y}(x,y) = j'_y(x,y) = -3x\cos(x^2-3xy).$$

$\dfrac{\partial^2 j}{\partial x^2}(x,y) = j''_{x^2}(x,y) = 2\cos(x^2-3xy) - (2x-3y)^2\sin(x^2-3xy).$

$\dfrac{\partial^2 j}{\partial y\partial x}(x,y) = j''_{yx}(x,y) = -3\cos(x^2-3xy) + (6x^2-9xy)\sin(x^2-3xy).$

$\dfrac{\partial^2 j}{\partial y^2}(x,y) = j''_{y^2}(x,y) = -9x^2\sin(x^2-3xy).$

$\dfrac{\partial^2 j}{\partial x\partial y}(x,y) = j''_{xy}(x,y) = -3\cos(x^2-3xy) + 3x(2x-3y)\sin(x^2-3xy).$

Correction 3

1. Une fonction f de $\mathbb{R}^n$ dans $\mathbb{R}$ est dite homogène de degré k si et seulement si pour tout réel λ et pour tout élément u de $\mathbb{R}^n$, on a $f(\lambda u) = \lambda^k f(u)$.
   Autrement dit, si on pose $u = (x_1, x_2, \ldots, x_n)$, on a :
   $f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k f(x_1, x_2, \ldots, x_n)$

2. Si f est une fonction homogène de degré k, alors pour tout élément $(x_1, x_2, \ldots, x_n)$ de $\mathbb{R}^n$, on a $x_1 f'_{x_1}(x_1,\ldots,x_n) + \ldots + x_n f'_{x_n}(x_1,\ldots,x_n) = kf(x_1,\ldots,x_n)$

3. $f(x,y) = \dfrac{2xy^3}{x+y}$

   (a) $f(\lambda x, \lambda y) = \dfrac{2(\lambda x)(\lambda y)^3}{(\lambda x)+(\lambda y)} = \dfrac{2\lambda^4 xy^3}{\lambda(x+y)} = \lambda^3\dfrac{2xy^3}{x+y} = \lambda^3 f(x,y)$
   La fonction f est homogène de degré 3.

   (b) $\dfrac{\partial f}{\partial x}(x,y) = f'_x(x,y) = \dfrac{(2y^3)(x+y) - 2xy^3(1)}{(x+y)^2} = \dfrac{2y^4}{(x+y)^2}$
   $\dfrac{\partial f}{\partial y}(x,y) = f'_y(x,y) = \dfrac{(6xy^2)(x+y) - 2xy^3(1)}{(x+y)^2} = \dfrac{6x^2y^2+4xy^3}{(x+y)^2}$

   (c) $xf'_x(x,y) + yf'_y(x,y) = x\dfrac{2y^4}{(x+y)^2} + y\dfrac{6x^2y^2+4xy^3}{(x+y)^2} = \dfrac{2xy^4}{(x+y)^2} + \dfrac{6x^2y^3+4xy^4}{(x+y)^2}$
   $= \dfrac{6x^2y^3+6xy^4}{(x+y)^2} = \dfrac{6xy^3(x+y)}{(x+y)^2} = 3\dfrac{2xy^3}{x+y} = 3f(x,y)$

   (d) $f'_x(\lambda x, \lambda y) = \dfrac{2(\lambda y)^4}{(\lambda x+\lambda y)^2} = \dfrac{2\lambda^4 y^4}{\lambda^2(x+y)^2} = \lambda^2\dfrac{2y^4}{(x+y)^2}$
   $= \lambda^2 f'_x(x,y)$
   $f'_y(\lambda x, \lambda y) = \dfrac{6(\lambda x)^2(\lambda y)^2 + 4(\lambda x)(\lambda y)^3}{(\lambda x+\lambda y)^2}$
   $= \dfrac{\lambda^4(6x^2y^2+4xy^3)}{\lambda^2(x+y)^2}$
   $= \lambda^2\dfrac{6x^2y^2+4xy^3}{(x+y)^2} = \lambda^2 f'_y(x,y)$
   Donc f'_x et f'_y sont homogènes de degré 2.
   Puisque f'_x et f'_y sont homogènes de degré 2, on a les relations suivantes :
   $x\dfrac{\partial f'_x}{\partial x}(x,y) + y\dfrac{\partial f'_x}{\partial y}(x,y) = 2f'_x(x,y)$

$x\dfrac{\partial f'_y}{\partial x}(x,y) + y\dfrac{\partial f'_y}{\partial y}(x,y) = 2f'_y(x,y)$

Autrement dit :

$x\dfrac{\partial^2 f}{\partial x^2}(x,y) + y\dfrac{\partial^2 f}{\partial y \partial x}(x,y) = 2\dfrac{\partial f}{\partial x}(x,y)$

$x\dfrac{\partial^2 f}{\partial x \partial y}(x,y) + y\dfrac{\partial^2 f}{\partial^2 y}(x,y) = 2\dfrac{\partial f}{\partial y}(x,y)$

Puisque f est homogène de degré 3, on a :

$x\dfrac{\partial f}{\partial x}(x,y) + y\dfrac{\partial f}{\partial y}(x,y) = 3f(x,y)$

Ou encore $6f(x,y) = x\left(2\dfrac{\partial f}{\partial x}(x,y)\right) + y\left(2\dfrac{\partial f}{\partial y}(x,y)\right)$

En remplaçant $\dfrac{\partial f}{\partial x}$ et $\dfrac{\partial f}{\partial y}$ par les valeurs précédentes :

$6f(x,y) = x\left(x\dfrac{\partial^2 f}{\partial x^2}(x,y) + y\dfrac{\partial^2 f}{\partial y \partial x}(x,y)\right) + y\left(x\dfrac{\partial^2 f}{\partial x \partial y}(x,y) + y\dfrac{\partial^2 f}{\partial^2 y}(x,y)\right)$.

Correction 4

Par définition, on a $E_{D/P_A} = \dfrac{\dfrac{\partial D}{\partial p_A} \times p_A}{D}$.

Ici $E_{D/P_A} = \dfrac{-0,3p_A^{-1,3}p_B^{0,1}p_C^{-0,4} \times p_A}{p_A^{-0,3}p_B^{0,1}p_C^{-0,4}} = -0,3$.

$E_{D/P_B} = \dfrac{\dfrac{\partial D}{\partial p_B} \times p_B}{D} = \dfrac{0,1p_A^{-0,3}p_B^{-0,9}p_C^{-0,4} \times p_B}{p_A^{-0,3}p_B^{0,1}p_C^{-0,4}} = 0,1$.

De même, $E_{D/P_C} = -0,4$.