Université de Picardie Jules Verne

UFR d'économie et de gestion

Mathématiques

Licence 1 - Semestre 2

Exercices d'entrainement **Series numériques** **Corrigés**

Correction 1

- On a $\lim\limits_{n\to+\infty} x^n = 0$. En effet, $x^n = e^{n\ln x}$ avec $\ln x < 0$.
- On a $\sum\limits_{k=0}^{n} x^k = \dfrac{1-x^{n+1}}{1-x}$ donc $\lim\limits_{n\to+\infty} \sum\limits_{k=0}^{n} x^k = \dfrac{1}{1-x}$.
- $\sum\limits_{k=0}^{n} kx^k = \dfrac{nx^{n+2}-(n+1)\,x^{n+1}+x}{(x-1)^2}$.

On a $\lim\limits_{n\to+\infty} nx^{n+2} = 0$ et $\lim\limits_{n\to+\infty} (n+1)\,x^{n+1} = 0$.

Donc $\sum\limits_{k=0}^{n} kx^k = \dfrac{x}{(1-x)^2}$.

- De même, on trouve $\lim\limits_{n\to+\infty} \sum\limits_{k=0}^{n} k^2x^k = \dfrac{x^2+x}{(1-x)^3}$.

$$\sum_{k=0}^{n}\left(3k^2-5k+2\right)\left(\frac{1}{4}\right)^k = 3\sum_{k=0}^{n}k^2\left(\frac{1}{4}\right)^k - 5\sum_{k=0}^{n}k\left(\frac{1}{4}\right)^k + 2\sum_{k=0}^{n}\left(\frac{1}{4}\right)^k$$

Donc $\lim\limits_{n\to+\infty}\sum\limits_{k=0}^{n}\left(3k^2-5k+2\right)\left(\frac{1}{4}\right)^k$

$$= \lim_{n\to+\infty}\left(3\sum_{k=0}^{n}k^2\left(\frac{1}{4}\right)^k - 5\sum_{k=0}^{n}k\left(\frac{1}{4}\right)^k + 2\sum_{k=0}^{n}\left(\frac{1}{4}\right)^k\right)$$

$= 3\lim\limits_{n\to+\infty} k^2\sum\limits_{k=0}^{n}\left(\frac{1}{4}\right)^k - 5\lim\limits_{n\to+\infty}\sum\limits_{k=0}^{n}k\left(\frac{1}{4}\right)^k + 2\lim\limits_{n\to+\infty}\sum\limits_{k=0}^{n}\left(\frac{1}{4}\right)^k = 3\times\dfrac{\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)}{\left(1-\frac{1}{4}\right)^3} - 5\times\dfrac{\frac{1}{4}}{\left(1-\frac{1}{4}\right)^2} + 2\times\dfrac{1}{1-\frac{1}{4}} = 3\times\dfrac{5}{\frac{27}{4}} - 5\times\frac{1}{\frac{9}{4}} + 2\times\dfrac{1}{\frac{3}{4}} = \dfrac{20}{9} - \dfrac{20}{9} + \dfrac{8}{3} = \dfrac{8}{3}$.

Correction 2

1. $\dfrac{1}{k\,(k+1)} = \dfrac{a}{k} + \dfrac{b}{k+1}$
   $\Leftrightarrow \dfrac{1}{k\,(k+1)} = \dfrac{a(k+1)}{k(k+1)} + \dfrac{bk}{k(k+1)}$
   $\Leftrightarrow \dfrac{1}{k\,(k+1)} = \dfrac{(a+b)k+a}{k(k+1)}$
   $\Leftrightarrow a+b=0$ et $a=1$
   $\Leftrightarrow$ a $=1$ et $b=-1$.

C'est-à-dire $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$.

2. $\displaystyle\sum_{k=1}^{n} \frac{1}{k+1} = \left(\sum_{k=0}^{n} \frac{1}{k+1}\right) - 1 = \left(\sum_{k=0}^{n-1} \frac{1}{k+1}\right) - 1 + \frac{1}{n+1} = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - 1 + \frac{1}{n+1}$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k} - \left[\left(\sum_{k=1}^{n} \frac{1}{k}\right) - 1 + \frac{1}{n+1}\right]$$
$$= 1 - \frac{1}{n+1}$$

3. $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$

Correction 3

$$\begin{aligned}\sum_{k=0}^{n} \frac{5}{2^k} &= 5\sum_{k=0}^{n} \frac{1}{2^k} \\ &= 5\sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \\ &= 5 \times \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \\ &= 5 \times \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} \\ &= 10 \times \left(1 - \left(\frac{1}{2}\right)^{n+1}\right).\end{aligned}$$

On en déduit :

$$10 \times \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) > 9,99$$

$\Leftrightarrow\ 1 - \left(\frac{1}{2}\right)^{n+1} > 0,999$

$\Leftrightarrow\ 1 - 0,999 > \left(\frac{1}{2}\right)^{n+1}$

$\Leftrightarrow\ \dfrac{1}{2^{n+1}} < 0,001$

$\Leftrightarrow\ 2^{n+1} > 1000$ on a $2^9 = 512$ et $2^{10} = 1024$

$\Leftrightarrow\ n + 1 \geq 10$

$\Leftrightarrow\ n \geq 9$

Correction 4

$$\sum_{k=0}^{n} (4k^2 + 8k - 2)\left(\frac{1}{3}\right)^k = \sum_{k=0}^{n} \left[4k^2\left(\frac{1}{3}\right)^k + 8k\left(\frac{1}{3}\right)^k - 2\left(\frac{1}{3}\right)^k\right]$$

$$= \sum_{k=0}^{n}\left[4k^2\left(\frac{1}{3}\right)^k\right] + \sum_{k=0}^{n}\left[8k\left(\frac{1}{3}\right)^k\right] - \sum_{k=0}^{n}\left[2\left(\frac{1}{3}\right)^k\right]$$
$$= 4\sum_{k=0}^{n} k^2\left(\frac{1}{3}\right)^k + 8\sum_{k=0}^{n} k\left(\frac{1}{3}\right)^k - 2\sum_{k=0}^{n}\left(\frac{1}{3}\right)^k$$

$$\lim_{n\to+\infty}\sum_{k=0}^{n}\left(4k^2+8k-2\right)\left(\frac{1}{3}\right)^k$$
$$= \lim_{n\to+\infty}\left(4\sum_{k=0}^{n} k^2\left(\frac{1}{3}\right)^k + 8\sum_{k=0}^{n} k\left(\frac{1}{3}\right)^k - 2\sum_{k=0}^{n}\left(\frac{1}{3}\right)^k\right)$$
$$= 4\times\frac{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)^3} + 8\times\frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} - 2\times\frac{1}{1-\frac{1}{3}}$$
$$= 4\times\frac{3}{2} + 8\times\frac{3}{4} - 2\times\frac{3}{2} = 6+6-3 = 9.$$

Correction 5

$$\sum_{i=0}^{n}(2x_i+3)^2 = \sum_{i=0}^{n}\left(4x_i^2+12x_i+9\right) = 4\sum_{i=0}^{n}x_i^2 + 12\sum_{i=0}^{n}x_i + \sum_{i=0}^{n}9$$
$$= 4\times 40 + 12\times 15 + 9\times(n+1)$$
$$= 349 + 9n$$